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A Contribution To m-Power Closed Groups

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ABSTRACT

The notion of (m-power closed) group was introduced Kappe et.al.We say that a group G is (m-power closed) if $\boldsymbol{G_m} = \{\boldsymbol{g^m}; \boldsymbol{g} \in \boldsymbol{G}\}$with fixed integer m is a subgroup of G.
In this paper we study how the properties of $\boldsymbol{G_m}$ effects on G.
We focus on the case that m is a prime and G is a finite group, we prove that G is solvable if and only if$\boldsymbol{G_m}$ is solvable under the previous conditions, and we tried to determine a sufficient condition for a group G to be (m-power closed). Also, we studied the special case when a group G is (m-power closed) and (n-power closed) with relatively prime n,m which we call a Monic group and we determine some interesting properties.

1. Introduction

Date palm is a commonly edible fruit and has been known widely by the society of Kingdom of Saudi Arabia (KSA) and other Gulf countries. In KSA, there might be more than five hundred cultivars, and

A group G is said to be (m-power closed) if we have $\boldsymbol{G_m} = \{\boldsymbol{g^m}; \boldsymbol{g} \in \boldsymbol{G}\}$ which is a subgroup of G.

The previous definition is equivalent to the condition $\forall x, y \in G \exists z \in G \text{ such that } x^m y^m = z^m$.

The class of (m-power closed) groups is quotient and direct products are closed, but not subgroup closed see [1].

We will denote to (m-closed group) by (m-group).

1.1. Lemma

Let G be an (m-group) then:

(a)$G_m \blacktriangleright G$ and G/G_m is (m-group)

(b)If H is a fully invariant (m-subgroup) of G then $H_m \blacktriangleright G$

Proof:(a) Let φ be a homomorphism on G and x be an arbitrary element in $G_m$then ∃ y∈G such x=y^m , $\varphi(x) = \varphi(y^m) = (\varphi(y))^m \in G_m$ so G_m is a fully invariant subgroup so $G_m \blacktriangleright G$ and $(G/G_m)_m = G_m/G_m = \{e\} \leq G/G_m$, the proof is complete.

(b)$\forall h^m \in H_m$ such h∈ H we have for a homomorphism φ on G that: $\varphi(h^m) = (\varphi(h))^m \in H_m$so H_m is fully invariant and then normal according to [2].

1.2. Definition

Let G be a group and H▶G, we say that H is (m-normal factor) of G if and only if the following condition is true: $\forall x, y \in G \exists z \in G\,;\, z^m x^m y^m \in H$, we denote that by $H \blacktriangleright_m G$.

1.3. Theorem

Let G be a group then:

(a)If $H \blacktriangleright_m G$ and K▶G then $HK \blacktriangleright_m G$

(b)If H▶G then G/H is an (m-group) if and only if $H \blacktriangleright_m G$

(c)If H▶G then $H \blacktriangleright_m G$ if and only if $G_m H \leq G$

Proof:(a) Obviously HK▶G, now suppose that x,y∈ G there is z∈ G such $z^m x^m y^m$ ∈H≤ HK so $HK \blacktriangleright_m G$

(b)Assume that G/H is an (m-group), let x,y be two arbitrary elements in G then xH,yH∈ G/H so $(xH)^m (yH)^m \in (G/H)_m$ that implies $x^m y^m H = z^m H$ for some z∈ G which means that $(z^{-1})^m x^m y^m \in H$ and $H \blacktriangleright_m G$

Conversely assume that $H \blacktriangleright_m G$, let xH,yH be two arbitrary elements in G/H , we have $x, y \in G$ so there is z∈ G such $z^m x^m y^m \in H$ that means $x^m y^m H = (z^{-1})^m H$ and G/H must be (m-group)

(c)suppose that $G_m H$ is a subgroup of G then for each x,y∈ G and $h_1, h_2 \in H$ we have : $(x^m h_1)(y^m h_2)$=$z^m h_3$; z∈ $G and h_3 \in H$ so, in addition to the normality of H we can write : $z^m h_3 = (x^m h_1)(y^m h_2) = x^m y^m ((h')_1 (h_2)$; $(h)'_1 \in H$

So $z^{-m} x^m y^m \in H$ and $H \blacktriangleright_m G$

Conversely suppose that $H \blacktriangleright_m G$ and x,y∈ G , $h_1, h_2 \in H$
$(x^m h_1)(y^m h_2)^{-1} = x^m y^{-m} (h)'_1 h_2$; $(h)'_1 \in H$

There is z∈ G such $(z^{-1})^m x^m y^{-m} = h \in H$ so $x^m y^{-m} = z^m h$and we get that $(x^m h_1)(y^m h_2)^{-1} = (z)^m h (h)'_1 h_2 \in G_m H$ this implies $G_m H \leq G$

1.4. Theorem

(a)If $N \blacktriangleright_m G$ and G is a semi direct product of N and H then H is (m-group)

(b)If H▶G and K▶G and G=HK then N=H∩K $\blacktriangleright_m G$ if and only if $N \blacktriangleright_m H$ and $N \blacktriangleright_m K$

Proof:(a) We have G=HN with H∩N=$\{e\}$ so G/N ≅H and H must be (m-group)

(b)Suppose that $N \blacktriangleright_m H$ and $N \blacktriangleright_m K$, considering that G/N≅H/N×K/N and H/N, K/N are (m-groups) then G/N is (m-group) thus $N \blacktriangleright_m G$

Conversely let $N \blacktriangleright_m G$ then G/N is (m-group) so that H/N , K/N are (m-groups) , that means $N \blacktriangleright_m H$ and $N \blacktriangleright_m K$

2. Finite (m-groups)

In this section we consider a finite group G and we denote to (m-group) with prime m by (m^*-group)

2.1. Lemma

Let G be an (m-group) then:

(a)If G is (n-group) and (d-group) then $G_m G_n = G_d$; d=gcd(n,m)

(b)If G is (n-group) with gcd(n,m)=1 then G=$G_m G_n$

(c)If gcd(m,$|G|$)=d then $G_m = G_d$

(d)If gcd(m,$|G|$)=1 then G=G_m

Proof:(a)there are two integers a,b such d=am+bn , $\forall x^d \in G_d then x^d = (x^a)^m (x^b)^n \in G_m G_n$ so $G_d \leq G_m G_n$.Now $\forall x^m \in G_m then x^m = (x^t)^d\,; m = td$ so $G_m \leq G_d$, according to the same argument we find that $G_n \leq G_d$ which means $G_m G_n \leq G_d$ and the proof is complete.

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(b)Holds directly from (a)
(c) see [3]
(d)Holds directly from (c)

2.2. Lemma

Let G be an (m-group) and an (n-group) ; gcd(n,m)=1 and let $|G| = mnq$; q$\in N$ then:
(a)If $G_m = G_n = \{e\}$ then G=$\{e\}$
(b)$G_n/(G_m \cap G_n) \cong$ G/G_m and $G_m/(G_n \cap G_m) \cong$G/G_n
(c)G/$(G_m \cap G_n) \cong$ G/G_m ×G/G_n
(d)$(G_m \cap G_n)_q = \{e\}$
(e)n/$|G_m|$ and m/$|G_n|$
Proof:
(a)there are two integers a,b such 1=an+bm , let x be an arbitrary element of G then :
x=$x^1 = x^{an}x^{bm} = e.e = e$ so G=$\{e\}$
(b)Holds by isomorphism theorem
(c)Holds directly from (b) and lemma 3.1
(d)Let H=$G_n \cap G_m$ then H$\leq G_n and H \leq G_m$ so $H_{nq} \leq G_{mnq} = \{e\} and H_{mq} \leq G_{mnq} = \{e\}$ thus $(H_q)_m = (H_q)_n = \{e\}$ .Following the first condition, we can see that $H_q = \{e\}$
(e)We can find at least one element x$\in G$ such that $x^n = e$ because n/$|G|$ thus $G_n \neq G$, $(G/G_n)_m$ =G_nG_m/G_n=G/G_n , using the previous argument we find that gcd (m,$\frac{|G|}{|G_n|}$)=1
, but m/$|G|$ so that m/$|G_n|$ .The second proposition can be proved by the same way.

2.3. Theorem

Let G be an (m^*-group) with m/$|G|$, let $|G|$=$m^{k1}p_2{}^{k2} \ldots . p_s{}^{ks}$; $p_i are\ distinct\ primes\ for\ each\ 2 \leq i \leq s$ then :
(a) $p_2{}^{k2} \ldots . p_s{}^{ks}/|G_m|$
(b) G/G_m is a (p-group) with m=p
(c) G is solvable if and only if G_m is solvable
Proof:
(a)for each primep_i the (p_i-Sylow) subgroup H_i has order $p_i{}^{ki}$ with gcd($p_i{}^{ki}$, m)=1
So $(H_i)_m = H_i \leq G_m$ then $p_i{}^{ki}/|G_m|$ for each i ,thus $p_2{}^{k2} \ldots . p_s{}^{ks}/|G_m|$
(b)$|G/G_m|$=m^k ; k$\leq k_1$ so G/G_m is a (p-group) with m=p
-we meant by (p-group) a group with order p^s ; s$\in N$ and p is prime-
(c)Assume that G_m is solvable then G/G_m is also solvable because it is a (p-group) according to [4] , this means that G is solvable, the converse is clear.

2.4. Theorem

Let G be an (m^*-group) with m/$|G|$ then:
(a) If G simple, then it is cyclic of order m
(b)If H►G then H/$(H \cap G_m)$ is a (p-group) with p=m
Proof:(a) We have m/$|G|$ so that G$\neq G_m$, but G_m►G so $G_m = \{e\}$ and G/G_m is a (p-group) and in this case G/G_m ≅G which means that G is a simple (p-group) then G is cyclic with order m
(b)Suppose that H►G then $G_m \cap H$►H and H/$(H \cap G_m) \cong G_mH/G_m \leq G/G_m$ so H/$(H \cap G_m)$ is a (p-group)

2.5. Remark

If we consider that the finite group G is (m^k-group) with $|G|$=$m^{k1}p_2{}^{k2} \ldots . p_s{}^{ks}$; m,$p_2, p_3 \ldots . p_s$ are distinct primes and k$\leq k_1$ then theorems (3.3) and (3.4) are still true.

3. Power central series

3.1. Definition

Let G be a group with center Z(G) we define the (m-center) of G by $Z_m(G) = (Z(G))_m$

3.2. Theorem

Let G be a group then G is (m-group) if and only if $G/Z_m(G)$ is (m-group)
Proof:
It is easy to see that $Z_m(G)$ is a characteristic subgroup so it is normal
If G is an (m-group), then $G/Z_m(G)$ is (m-group). Conversely suppose that $G/Z_m(G)$ is (m-group) and x , y be two arbitrary elements of G then there is $z \in G$ such $z^{-m}x^my^m = k^m \in Z_m(G)$ thus $x^my^m = z^mk^m$, we have $k \in Z(G)$ which implies that $z^mk^m = (zk)^m$ so $x^my^m = (zk)^m$ and G is (m-group)

3.3. Definition

Let G be a group we define $Z_i^m(G)$ to be the subgroup of G such that $Z_i^m(G)/Z_{i-1}^m(G) = Z_m(G/Z_{i-1}^m(G))$ with $Z_0^m(G) = G$
By the previous definition, we get the series $\{e\} \leq Z_1^m(G) \leq Z_2^m(G) \leq \cdots \leq Z_i^m(G) \leq \cdots$

3.4. Theorem

Let G be a group then:
(a) G is (m_group) if and only if there is an integer i such $Z_i^m(G)$►$_mG$
(b) if there is an integer i such $Z_i^m(G) = G$ then G is (m-group)
(c) if G is finite and there is an integer i such gcd $(|G/Z_i^m(G)|, m) = 1$ then G is (m_group)
Proof:
(a) Assume that there is an integer i such $Z_i^m(G)$►$_mG$ then $G/Z_i^m(G) \cong (G/Z_{i-1}^m(G))/Z_m(G/Z_{i-1}^m(G))$ is (m_group) so that $G/Z_{i-1}^m(G)$ is (m_group) , by the same argument we get $G/Z_m(G)$ is (m-group) so G is (m_group) by theorem 2.3
(b) Assume that there is an integer i such $Z_i^m(G) = G$ then $Z_i^m(G)$►$_mG$ so G is (m_group)
(c) Assume that G is finite and there is an integer i such gcd $(|G/Z_i^m(G)|, m) = 1$ then $(G/Z_i^m(G))_m = G/Z_i^m(G)$ so $Z_i^m(G)$►$_mG$ and G is (m_group)

4. Monic groups

4.1. Definition

Let G be a finite (m-group) with m/$|G|$ then we say that it is a monic group if and only if G is an (n-group) with n/$|G|$ and gcd (n,m)=1

4.2. Lemma

Let G be a monic group then:
(a) G=G_mG_n
(b) The homomorphic image of G is also monic
Proof:
(a) Holds from lemma 1.2
(b) Since the homomorphic image of (m-group) is also (m-group) the proof is complete

4.3. Lemma

Let G be a group and H►G then G/H is monic if and only if H►$_mG and H$►$_nG$
Proof:
Since G/H is (m-group) if and only if H►$_mG$ then the proof is complete

4.4. Theorem

Let G be a monic group then:
(a) G is solvable if and only if G_m , G_n are solvable
(b) If G_m , G_n are nilpotent groups then G is nilpotent
(c) If G_m , G_n have an abelian automorphism group then G has an abelian automorphism group
Proof: (a) Since G=G_mG_n then G is solvable if and only if G_m , G_n are solvable
(b) Suppose that G_m , G_n are nilpotent groups then G_mG_n is nilpotent so G is.
(c) Suppose that G_m , G_n have an abelian automorphism group , $G_m and G_n$ are characteristic subgroups of G then for each $f \in aut(G) we have f \in aut(G_m) \cap aut(G_n)$, now there are two integers a,b such am+bn=1 and $\forall f, g \in aut(G) and for an abitrary element x \in G we have that f \circ g(x) = f \circ g(x^{am}x^{bn}) = f \circ g((x^a)^m) f \circ g((x^b)^n) = g \circ f((x^a)^m) g \circ f((x^b)^n) = g \circ f(x^{am}x^{bn}) = g \circ f(x)$ so G has an abelian automorphism group

4.5. Theorem

Let G be a monic group then G is cyclic if and only if G_m , G_n are cyclic
Proof: If G is cyclic then it is monic with cyclic G_m , G_n . Conversely suppose that G_m , G_n are cyclic then they are nilpotent so G is nilpotent and G is a direct product of its Sylow subgroups, let P_i be the (i-th) Sylow subgroup of this product with orderp^s then P_i is (m-cyclic) and (n-cyclic) , because p is a prime we find that gcd (m,p)=1 or gcd (n,p)=1 , without affecting the generality we assume that gcd (m,p)=1

so $(P_i)_m = P_i$ and P_i must be cyclic. By cyclicity of G_m, G_n we find that they have an abelian automorphism group so G is a direct product of cyclic groups with abelian automorphism group then G must be cyclic.

4.6. Theorem

Let G be a monic group then G is abelian if and only if G_m , G_n are abelian

Proof: If G is abelian then it is monic with abelian G_m , G_n . Conversely suppose that G_m , G_n are abelian then they are nilpotent so G is nilpotent and G is a direct product of its Sylow subgroups , let P_i be the (i-th) Sylow subgroup of this product with orderp^s then P_i is (m-abelian) and (n-abelian) , because p is a prime we find that gcd (m,p)=1 or gcd (n,p)=1 , without affecting the generality we assume that gcd (m,p)=1 so $(P_i)_m = P_i$ and P_i must be abelian. So G is a direct product of abelian groups so G is abelian

4.7. Theorem

Let G be a finite nilpotent group then G is monic [5]

Proof: Assume that G is nilpotent then $G = P_1 \times P_2 \times \dots . P_n$ where $P_i isaSylowsubgroupwithorderp_i{}^{k_i}$, we put m=$p_1{}^{k_1}$ and n= $p_2{}^{k_2}$ then $G_m = P_2 \times \dots . P_n$ and $G_n = P_1 \times P_3 \times \dots . \times P_n$ so G is a monic group since gcd(n,m)=1

[6]

4.8. Theorem

The direct product of two monic groups is again a monic group.

Proof: Holds directly from theorem (1.2)

5. Conclusions

In this article, we have studied m-groups and determined the sufficient condition of a group G to be an m-group. Also, we have defined and studied Monic groups in particular**.**.

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